# Lecture 10. Binomial Trees II

EF4321. Derivatives and Risk Management

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10/11/12 November 2021

# Lecture Outline

- Risk-Neutral Valuation vs. DCF
- Two-Step Binomial Tree
- N-Step Binomial Tree
- Advanced Topics in Binomial Models
  - American Options
  - Determination of $u$ and $d$

# Risk-Neutral Valuation vs. DCF

- The option price from the risk-neutral valuation is **the same** as the price from the DCF.
- If required return on stock $\alpha$ is higher than the risk-free rate $r$, it follows that $p < p^*$.
- It implies that in the risk-neutral valuation, we amplify the probability of a bad outcome for stock investors.
- We interpret this as the probability being modified to incorporate investors' risk-aversion.

# Risk-Neutral Valuation

- Risk-neutral valuation is a **probabilistic interpretation** of the option pricing formula obtained from the replicating portfolio. Here, we price options as if ...
  1. stock price follows the risk-neutral probability.
  2. future option payoffs are discounted at the risk-free rate.

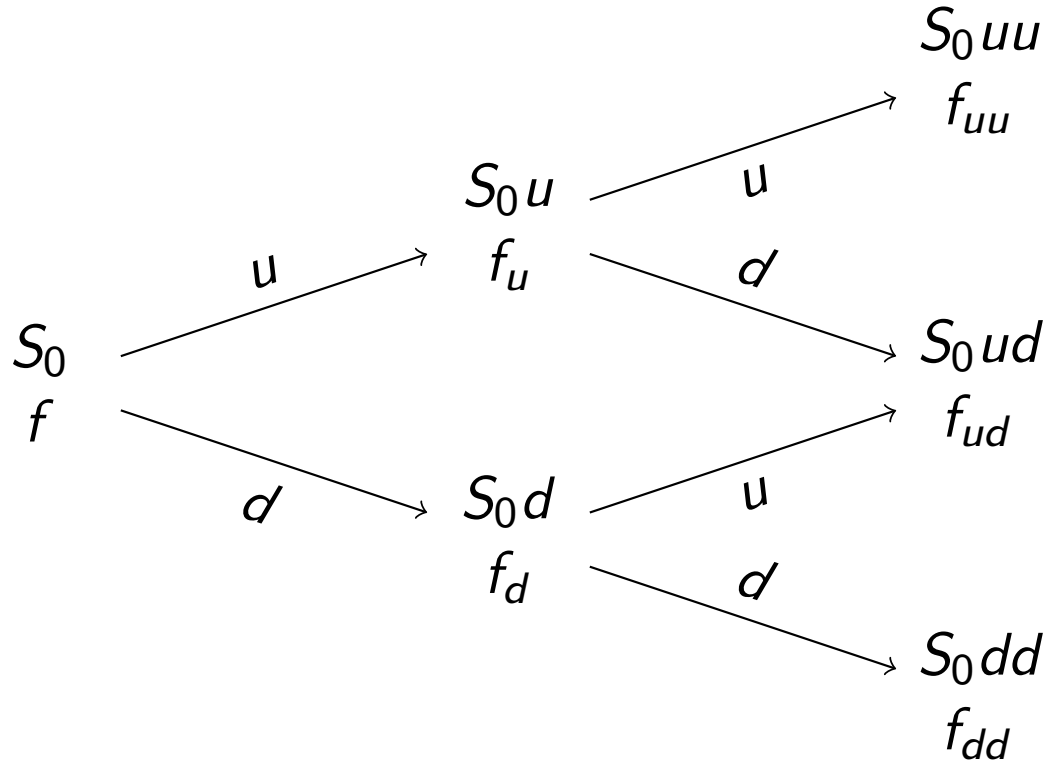
- This interpretation is also useful for multi-step binomial models.

# Two-Step Binomial Models

# Two-Step Binomial Model

- The current stock price is $S_0$ and may go up by $u$ or down by $d$ in a time step. Each time step is $\Delta t$ and the risk-free interest rate is $r$ per annum.

- A European call option has the strike price of $K$ and expires in two steps. What is the option price?



# Two-Step Binomial Model

- We start at the option expiration date and find the option payoff at each stock price then.

- At $T = \Delta t$, each price and the following prices can be seen as one-step binomial tree. Thus, we can use the pricing formula of one-step models.

$$\begin{cases} f_u &= e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right] \\ f_d &= e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right] \end{cases}$$

  where $p = \frac{e^{r\Delta t} - d}{u - d}$.

# Two-Step Binomial Model

- At $T = 0$,

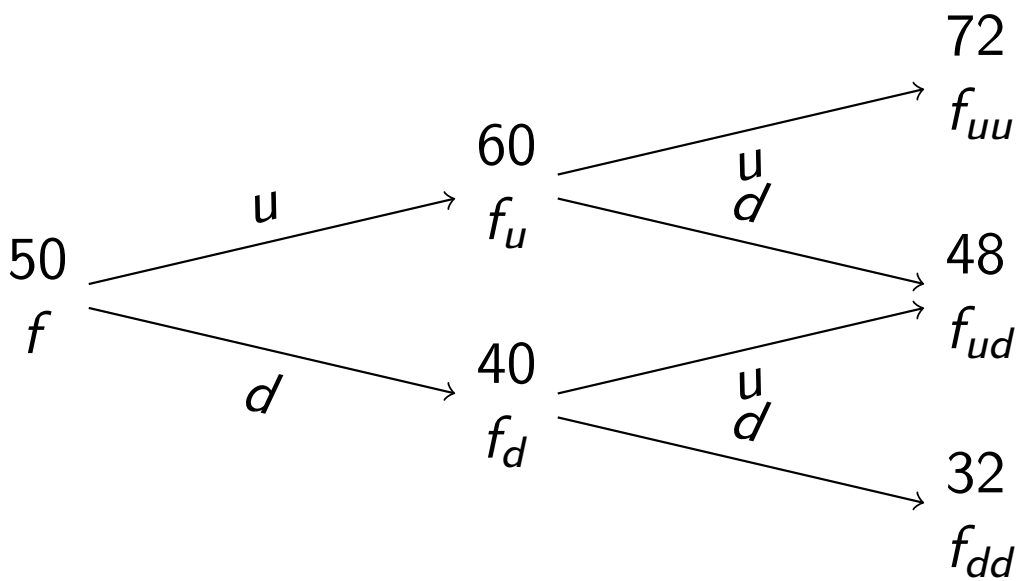
$$
\begin{aligned}
f_0 &= e^{-r\Delta t}\left[pf_u + (1-p)f_d\right] \\
&= e^{-r\Delta t}\left[p\left(e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right]\right) + (1-p)\left(e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right]\right)\right] \\
&= e^{-2r\Delta t}\left[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right]
\end{aligned}
$$

- This is consistent with the probabilistic interpretation of $p$.
  - In risk-neutral valuation, $p^2$, $2p(1-p)$, and $(1-p)^2$ are probabilities of reaching top, middle, and bottom final nodes.

# Two-Step Binomial Model - Put Option

- To price a put option, we use put payoffs at the option expiration. The rest of calculation is the same as the call valuation.

Q. Consider a 2-year European put with $K$=\$52 on a stock with $S_0$=\$50. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



# N-Step Binomial Models

# N-Step Binomial Model

- Suppose that there are $N$ time steps until the option maturity and each time step is $\Delta t$.

- The risk-neutral probability of an increase in stock price during each step is $p(= \frac{e^{r\Delta t} - d}{u - d})$.

- There are $N + 1$ nodes at the expiration. Let node $j$ denote the final stock price when the price moves upward $j$ times and downward $N - j$ times. There, the final stock price would be

$$S_0 u^j d^{N-j},$$

where $j = 0, 1, ..., N$.

# N-Step Binomial Model

- To determine the price of an European option, we need the probability of reaching each node at the expiration.

- The probability of reaching the node $j$ is

$$\binom{N}{j} p^j (1-p)^{N-j}$$

  - There are multiple paths leading to the node $j$. The number of the paths is $\binom{N}{j}$, which is $j$-combinations from a set of $N$ elements.
  - How to calculate $\binom{N}{j}$?
    - In algebra, $\binom{N}{j} = \frac{N!}{j!(N-j)!}$.
    - In Excel, use "combin(N,j)".

# N-Step Binomial Model

- For each node, the probability and option payoff is as follows:

| No. of up | No. of down | Probability | Stock price at $T$ | Option payoff |
|---|---|---|---|---|
| $N$ | 0 | $p^N(1-p)^0$ | $S_0u^Nd^0$ | $f_N$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $j$ | $N-j$ | $\binom{N}{j}p^j(1-p)^{N-j}$ | $S_0u^jd^{N-j}$ | $f_j$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 1 | $N-1$ | $\binom{N}{1}p^1(1-p)^{N-1}$ | $S_0u^1d^{N-1}$ | $f_1$ |
| 0 | $N$ | $p^0(1-p)^N$ | $S_0u^0d^N$ | $f_0$ |

- The price of European option is then

$$e^{-r(N\Delta t)}\sum_{j=0}^{N}\binom{N}{j}p^j(1-p)^{N-j}f_j$$

where $f_j$ is the option payoff at node $j$.

# N-Step Binomial Model

Q. Consider a 3-year European call with $K$=\$30 on a stock with $S_0$=\$30. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 10%. The risk-free interest rate is 5%. What is the price of the call option?

# N-Step Binomial Model

Q. Consider a 3-year European call with $K$=\$30 on a stock with $S_0$=\$30. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 10%. The risk-free interest rate is 5%. What is the price of the call option?

**Answer:** First, the option payoffs at each of 4 nodes are

$$f_0 = \max(30(1.1)^0(0.9)^3 - 30, 0) = 0$$
$$f_1 = \max(30(1.1)^1(0.9)^2 - 30, 0) = 0$$
$$f_2 = \max(30(1.1)^2(0.9)^1 - 30, 0) = 2.67$$
$$f_3 = \max(30(1.1)^3(0.9)^0 - 30, 0) = 9.93.$$

The risk-neutral probability is $p = \frac{e^{0.05\times 1} - 0.9}{1.1 - 0.9} = 0.756$. Then, the option price is

$$e^{-0.05\times 3} \sum_{j=0}^{3} \binom{N}{j} (0.756)^j (1 - 0.756)^{N-j} f_j = 4.66$$

# Pricing American Options

# American Options

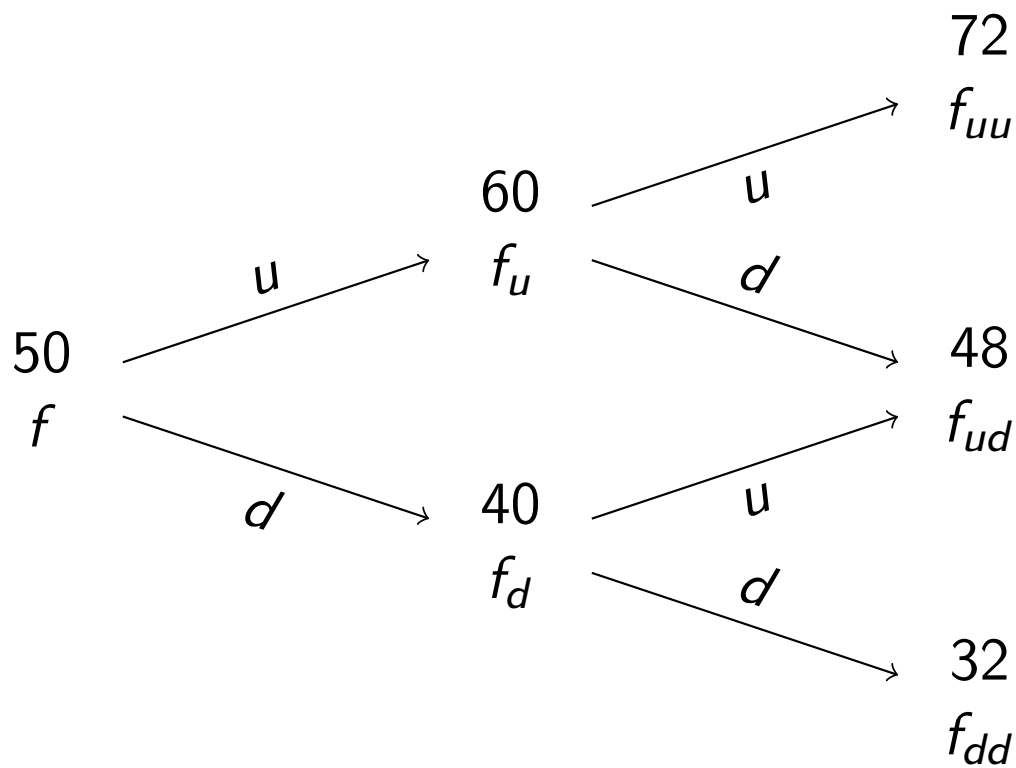
- In pricing American options, we should consider the possible early exercise of the options.
- In a similar way to pricing European options, we build a binomial tree of stock price. Then, we start from final nodes and proceed backward.
- However, the option value at each node becomes the maximum of
  1. the option value when delaying the exercise,

$$f = e^{-r\Delta t}\left[pf_u + (1-p)f_d\right]$$

  2. the payoff when exercising now

# American Options

e.g. Consider a 2-year American put with $K$=\$52 on a stock with $S_0$=\$50. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



# American Options

- The risk-neutral probability is

$$p = \frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = 0.6282.$$

- At final nodes, option payoffs are

$$\begin{cases} f_{uu} &= \max(52 - 72, 0) &= 0 \\ f_{ud} &= \max(52 - 48, 0) &= 4 \\ f_{dd} &= \max(52 - 32, 0) &= 20 \end{cases}$$

# American Options

- At top node in $T = 1$, the option price is

$$f_u = \max\left( \underbrace{e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right]}_{\text{value if delay}}, \quad \underbrace{52 - 60}_{\text{value if exercise}} \right) = \$1.415.$$

  At bottom node in $T = 1$, the option price is

$$f_d = \max\left(e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right], 52 - 40\right) = \$12.$$

- At the initial node, the option price is

$$f = \max\left(e^{-r\Delta t}\left[pf_u + (1-p)f_d\right], 52 - 50\right) = \$5.090.$$

# Determining $u$ and $d$

# Determining $u$ and $d$

- We have studied how to price options when $u$ and $d$ are given in binomial trees.
- If they are not given, how can we determine $u$ and $d$?
- For this determination, we focus on the volatility of underlying asset.
  - The volatility $\sigma$ is the standard deviation of yearly returns on the stock.
- The basic idea is to choose $u$ and $d$ such that the volatility in the binomial tree matches the volatility we see in data.
- First, how can we measure the volatility from data?

# Determining $u$ and $d$ - Estimating Volatility

- Suppose that we have data of returns over $\Delta t$ horizon, not yearly return.

- The standard deviation of $r_{\Delta t}$ can be related to the volatility as follows:

$$\text{Std.Dev.}\left(r_{\Delta t}\right) = \sigma\sqrt{\Delta t}.$$

  where $\sigma$ is the the standard deviation of $r_{\text{1-year}}$.

# Determining $u$ and $d$ - Estimating Volatility

- Suppose that we have the following historical prices of stock.

| Date | Price | Return |
|---|---|---|
| 0 | $P_0$ | |
| 1 | $P_1$ | $r_1 = \ln(P_1/P_0)$ |
| 2 | $P_2$ | $r_2 = \ln(P_2/P_1)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| T | $P_T$ | $r_T = \ln(P_T/P_{T-1})$ |

- Using these daily returns $(r_1, r_2, \cdots, r_T)$, we calculate the standard deviation Std.Dev $(r_{\text{1-day}})$.

- Then the volatility $\sigma$ can be found as follows:

$$\text{Std.Dev}\,(r_{\text{1-day}}) = \sigma\sqrt{\frac{1}{365}}.$$

# Determining $u$ and $d$

- Once the volatility $\sigma$ is obtained from data, we want to construct a binomial tree such that returns in the tree have the same volatility.
  - This means that the return over one step ($= \Delta t$) should have the standard deviation of $\sigma\sqrt{\Delta t}$.

- It turns out that we can achieve this by choosing

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

  in the binomial tree.

- Let's prove that under this choice of $u$ and $d$, the return in the binomial trees has the standard deviation of $\sigma\sqrt{\Delta t}$.

# Determining $u$ and $d$

- Suppose that the required return on the stock in the real world is $\alpha$ per annum. Then, the real probability $p^* = \frac{e^{\alpha \Delta t} - d}{u - d}$.
- Using this real probability, we can compute the variance of return $\text{Var}(r)$.
- We want to show that $\text{Var}(r)$ equals $\sigma^2 \Delta t$ under this particular choice of $u$ and $d$.

$$\begin{aligned}
\text{Var}(r) &= E(r^2) - [E(r)]^2 \\
&= p^* u^2 + (1 - p^*) d^2 - \left[e^{\alpha \Delta t}\right]^2 \\
&= p^*(u^2 - d^2) + d^2 - e^{2\alpha \Delta t} \\
&= \ldots
\end{aligned}$$

# Determining $u$ and $d$

$$\begin{aligned}
\mathrm{Var}(r) &= \dots \\
&= \frac{e^{\alpha\Delta t} - d}{u - d}(u + d)(u - d) + d^2 - e^{2\alpha\Delta t} \\
&= e^{\alpha\Delta t}(u + d) - d(u + d) + d^2 - e^{2\alpha\Delta t} \\
&= (u + d)e^{\alpha\Delta t} - ud - e^{2\alpha\Delta t}
\end{aligned}$$

- Now, let's plug in $u = e^{\sigma\sqrt{\Delta t}}$ and $u = e^{-\sigma\sqrt{\Delta t}}$.

$$\begin{aligned}
\mathrm{Var}(r) &= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{\alpha\Delta t} - 1 - e^{2\alpha\Delta t} \\
&= \dots ?
\end{aligned}$$

# Determining $u$ and $d$ - Math Review

- Taylor series
  - A function $f(x)$ can be expressed as a sum of polynomials:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

  - A Taylor series of $e^x$ is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

  - When $x$ is small, we can ignore higher-order terms.

# Determining $u$ and $d$

- Applying the Taylor series to the exponential terms, we can simplify the variance.

- Here, we assume that $\Delta t$ is very small, so we ignore $\Delta t^{3/2}$, $\Delta t^2$, and higher powers.
  - For instance, $e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t} + \frac{\left(\sigma\sqrt{\Delta t}\right)^2}{2}$.

- Then, the variance becomes

$$\begin{aligned}
\mathrm{Var}(r) &= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{\alpha\Delta t} - 1 - e^{2\alpha\Delta t} \\
&\approx (2 + \sigma\sqrt{\Delta t} - \sigma\sqrt{\Delta t} + \sigma^2\Delta t)(1 + \alpha\Delta t) - 1 - (1 + 2\alpha\Delta t) \\
&\approx \sigma^2\Delta t
\end{aligned}$$

# Things To Do

- Read the textbook chapters 13.3 - 13.5, 13.7 - 13.8
- Assignment 7 (due Friday, 19 November at 11 pm)